The Factor and Rational Root Theorem

The Factor Theorem

The factor theorem states that(x-a) is a factor of a polynomial P(x) if and only if P(a)=0. In other words:

- If(x-a) is a factor then, P(a)=0
- If P(a)=0, then (x-a) is a factor of a polynomial P(x).

Extended version of the factor theorem: $(bx-a)$ is a factor of a polynomial P(x) if $P\left(\frac{a}{b}\right)=0$

For example,

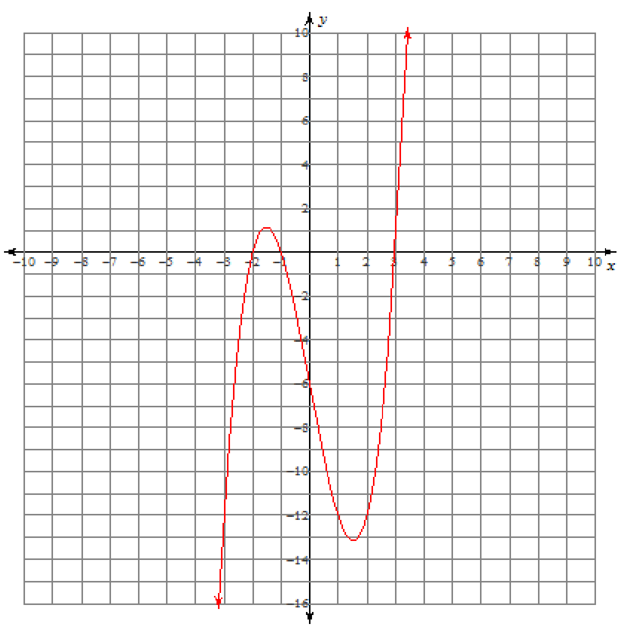
Given the polynomial, $P(x)=x^3-7x-6$, determine if (x-1) and (x+2) are factors by calculating P(1) and P(-2).

$$P(x)=x^3-7x-6$$
$$P(1)=1^3-7(1)-6$$
$$P(1)=1-7-6$$
$$P(1)=-12$$

Since P(1)=-12, then P(x) is not divisible by (x-1). Thus, (x-1) is not a factor of P(x).

$$P(x)=x^3-7x-6$$
$$P(-2)=(-2)^3-7(-2)-6$$
$$P(-2)=-8+14-6$$
$$P(-2)=0$$

Since P(-2)=0, then P(x) is divisible by (x+2). Thus, (x+2) is a factor of P(x).



The zeros of the polynomial function $P(x)=x^3-7x-6$ are related to the factors of the polynomial. The graph shows that the zeros of the function, or the x-intercepts of the graph are at

x=-2, x=-1 and x= 3.

The corresponding factors of the polynomial are: (x+2),(x+1), and (x-3).

Example 1: Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 + 4x^2 + x - 6$? Justify your answers.

a. x – 1 b. x – 2 c. x + 2 d. x + 3

Solution:

$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x – 1)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x – 2)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x +2)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x + 3)
(x-1) is / is not a factor of $P(x) = x^3 + 4x^2 + x - 6$	(x-2) is / is not a factor of $P(x) = x^3 + 4x^2 + x - 6$	(x+2) is / is not a factor of $P(x) = x^3 + 4x^2 + x - 6$	(x+3) is / is not a factor of $P(x) = x^3 + 4x^2 + x - 6$

Rational Zero (or Rational Root) Theorem

Given a polynomial function with integer coefficients, every rational zero will be of the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients then every rational zero of f(x) has the following form $\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$.

The Rational Root Theorem is a handy way of obtaining a list of useful first guesses when you are trying to find the zeroes (roots) of a polynomial. **Given a polynomial with integer coefficients, the possible zeroes are found by listing the factors of the constant (last) term over the factors of the leading coefficient, thus forming a list of fractions.** This listing gives you a list of *potential* rational (fractional) roots to test.

The Rational Zeros Theorem does *not* give you the zeroes. It does not say what the zeroes definitely will be. It only gives you a list of relatively easy and "nice" numbers to *try* in the polynomial. Most of these possible zeroes will turn out *not* actually to be zeroes!

For example:

Consider the polynomial function $f(x) = 6x^2 + 5x - 4$. The easiest method for obtaining the zeroes of this function is to set $f(x) = 0$, factor, and solve for x.

$0 = 6x^2 + 5x - 4 \rightarrow 0 = (3x+4)(2x-1) \rightarrow x = \frac{-4}{3}$ or x= $\frac{1}{2}$

However, to illustrate how to find the possible zeroes using the rational zero theorem, the theorem states that, for this polynomial, any rational zero must have a factor of -4 (________________) in the numerator and a factor of 6 (__________________________) in the denominator:

p: factors of -4= $\pm1, \pm2, \pm4$ q: factors of 6= $\pm1, \pm2, \pm3, \pm6$

The possibilities of $\frac{p}{q}$, in simplest form, are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm2, \pm\frac{2}{3}, \pm4, \pm\frac{4}{3}$

These values can be tested, using direct substitution or by using synthetic or long division, to see if the remainder is equal to zero. If it is, then $\frac{p}{q}$ is a zero of the function.

The rational root theorem is not recommended for finding the zeroes of a quadratic function, however, it is often used to find the zeroes of higher degree polynomial functions.

Example 2: Use the Rational Zero Theorem to Determine One of the Factors of a Polynomial with a Leading Coefficient of 1

For the polynomial function $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$:

a. Find all possible rational *roots* of P(x).
b. Determine one of the factors of P(x).

Solution:

a. All possible rational roots of P(x) must have a factor of ___________(constant term) in the numerator and a factor of ___________(leading coefficient) in the denominator.

p: factors of 12 = _______, _______, _______, _______, _______, _______
q: factors of 1 + _______

The Rational Zeros Theorem says that the possible zeroes are at:

$$\frac{p}{q} = \frac{\pm\_\_\_, \pm\_\_\_\_, \pm\_\_\_\_, \pm\_\_\_\_, \pm\_\_\_\_, \pm\_\_\_}{\pm1} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

b. Now determine one of the actual roots of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$

Since P(________) = 0 , then _______________is a factor of P(x).

Example 3: Use the Rational Zero Theorem to Determine One of the Factors of a Polynomial with Leading Coefficient Not Equal to 1.

For the polynomial function $P(x) = 2x^3 - 11x^2 + 12x + 9$:

a. Find all the possible rational roots of P(x).
b. Determine one of the factors of P(x).

Solution:

a. All possible rational roots of P(x) must have a factor of ______ (constant term) in the numerator and a factor of _______(leading coefficient)in the denominator.

The Rational Zeros Theorem says that the possible zeroes are at:

$$\frac{p}{q} = \frac{\pm\_\_\_, \pm\_\_\_\_, \pm\_\_\_\_\_}{\pm\_\_\_\_\_, \pm\_\_\_\_\_} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

b. Now determine one of the actual roots of $P(x) = 2x^3 - 11x^2 + 12x + 9$.

Since P(_______) = 0, _______________is a factor of P(x).

Example 4: Factor Using the Factor Theorem

a. Factor $2x^3 - 3x^2 - 11x + 6$ fully.
b. Use the factors of the polynomial expression to determine the zeros of the corresponding polynomial function.

Solution:

a. Let $P(x) = 2x^3 - 3x^2 - 11x + 6$

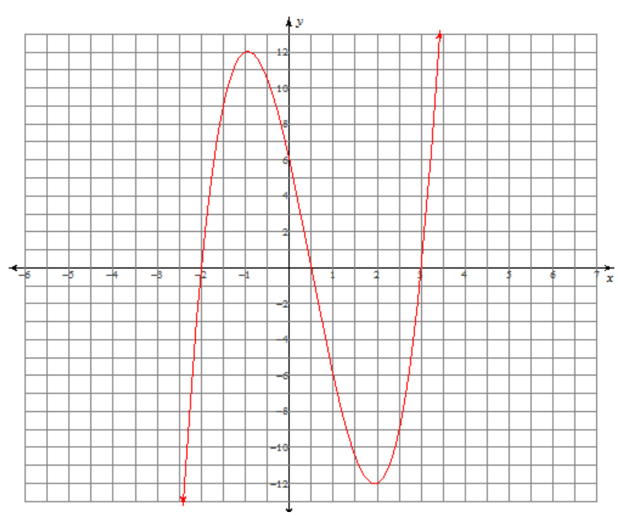
- Use the rational root theorem and factor theorems to determine one of the roots of the polynomial.

- Since P(   )=0, ________ is a factor of P(x).

- The remaining factor(s) can be determined by first using long or synthetic division and then following with specialized factoring strategies.

- Factors of the polynomial expression $P(x) = 2x^3 - 3x^2 - 11x + 6$ are __________, __________, __________.

b. The zeros of $P(x) = 2x^3 - 3x^2 - 11x + 6$ are ________, _________, and ________.



Example 5: Factor Higher-Degree Polynomials

Completely factor the polynomial $x^4+3x^3-7x^2-27x-18$.

Solution:

Let $P(x)=x^4+3x^3-7x^2-27x-18$

- Use the rational root and factor theorems to find one of the roots of the polynomial.

- Since P(   )=0, ________________________is a factor of P(x).
- The next factor can be determined by using long or synthetic division.

The remaining factor is _________________________________

- Let f(x)= ____
- Factor using one of the specialized factoring strategies, if possible. Or, use the rational root and factor theorems with synthetic or long division to find the remaining factors of the polynomial.

- Combine all the factors to write the fully factored form:
  $x^4+3x^3-7x^2-27x-18=$ ____

Example 6: Solve Problems Involving Polynomial Expressions

An artist creates a carving from a block of soapstone. The soapstone is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 6x^3 + 25x^2 + 2x - 8$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the block of soapstone?

Solution: