

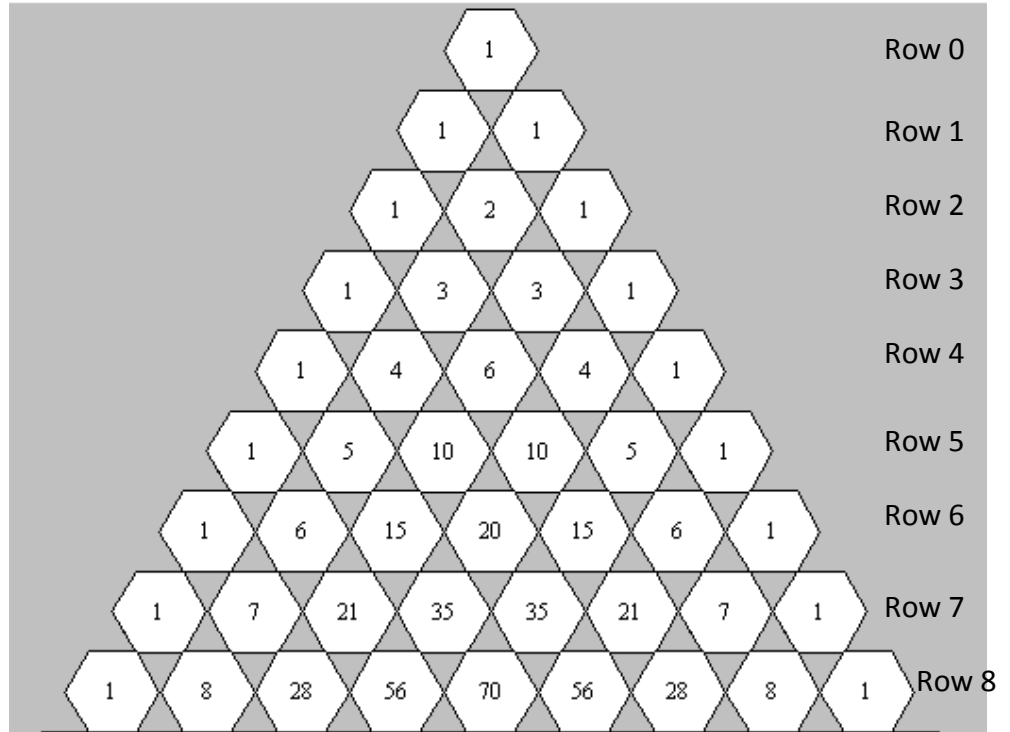
# The Binomial Theorem

## PASCAL'S TRIANGLE:

Pascal's triangle is a triangular arrangement of numbers constructed in such a way that every number in the interior of the triangle is the sum of the two numbers directly above it. It has many applications, including determining the coefficients of the expansion of a binomial expression or finding the number of ways to choose  $m$  objects from a group of  $n$  indistinguishable objects.

### Investigate Pascal's Triangle

1. Examine Pascal's Triangle. Identify at least 3 patterns.



2. One row of Pascal's triangle is

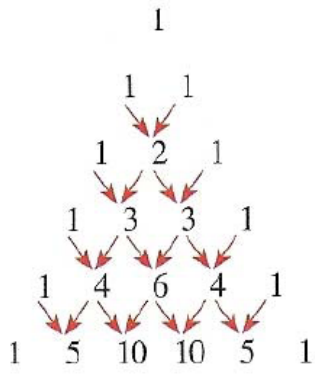
1    12    66    220    495    792    924    792    495    220    66    12    1

- a. What row would this be in Pascal's triangle?
- b. Determine the next row in the triangle.
- c. Determine the previous row.

3. Determine the sum of the numbers in each horizontal row. What pattern do you see?

4. Each number in Pascal's Triangle can be written as a combination using the notation  ${}_nC_r$ , where  $n$  is the number of objects in the set and  $r$  is the number selected.

$n = 0$ (0th row)	${}_0C_0$	1
$n = 1$ (1st row)	${}_1C_0$ ${}_1C_1$	1 1
$n = 2$ (2nd row)	${}_2C_0$ ${}_2C_1$ ${}_2C_2$	1 2 1
$n = 3$ (3rd row)	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	1 3 3 1
$n = 4$ (4th row)	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	1 4 6 4 1
$n = 5$ (5th row)	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	1 5 10 10 5 1



Evaluate each of the combinations in the 3<sup>rd</sup> row of the combinations triangle to confirm the values you obtain are those found in the 3<sup>rd</sup> row of Pascal's triangle.

5. Using combinations determine each of the following:
- a. the third element in the ninth row
  - b. the seventh element in the fifteenth row
  - c. the first five elements in row 20 of Pascal's Triangle.

## BINOMIAL EXPANSION

A binomial expression is the sum, or difference, of two terms. For example,  $x + 1$ ,  $3x + 2y$ ,  $a - b$  are all binomial expressions.

You are already familiar with the process by which to expand a binomial such as  $(x + 1)^2$  or  $(x + 1)^3$

$$\begin{aligned}(x + 1)^2 &= (x + 1)(x + 1) \\ &= x^2 + x + x + 1 \\ &= x^2 + 2x + 1\end{aligned}$$

$$\begin{aligned}(x + 1)^3 &= (x + 1)(x + 1)^2 \\ &= (x + 1)(x^2 + 2x + 1) \\ &= x^3 + 2x^2 + x + x^2 + 2x + 1 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

However, if you want to determine  $(x + 1)^6$  it is very cumbersome to do this by repeatedly multiplying  $x + 1$  by itself.

### EXPLORING $(x + 1)^n$

Pascal's triangle, can be used to determine the coefficient of each term in the binomial expansion of  $(x + 1)^n$ . Study the table to the right.

Power	Binomial Expansion	Pascal's Triangle
2	$(x + 1)^2 = 1x^2 + 2x + 1$	1, 2, 1
3	$(x + 1)^3 = 1x^3 + 3x^2 + 3x + 1$	1, 3, 3, 1
4	$(x + 1)^4 = 1x^4 + 4x^3 + 6x^2 + 4x + 1$	1, 4, 6, 4, 1
	... etc ...	

- For each expansion, what happens to the exponent as you move through each term from left to right?
- For each expansion, how are the coefficients related to Pascal's triangle?
- Compare the number of terms in each line of the expansion to the exponent the binomial is being raised to.
- Expand  $(x + 1)^6$ .

**EXPLORING**  $(x + y)^n$ 

Study the table to the right.

$$(x + y)^0 = 1 \longleftarrow \text{1 term}$$

$$(x + y)^1 = x + y \longleftarrow \text{2 terms}$$

$$(x + y)^2 = x^2 + 2xy + y^2 \longleftarrow \text{3 terms}$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \longleftarrow \text{4 terms}$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \longleftarrow \text{5 terms}$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \longleftarrow \text{6 terms}$$

- What happens to the exponents on  $x$  as you move through each term from left to right?
- What happens to the exponents on  $y$  as you move through each term from left to right?
- Compare the number of terms in each line of the expansion to the exponent the binomial is being raised to.
- For the row  $(x + y)^5$ , determine the degree of each term in the expansion. Compare this to another row.
- For each expansion, how are the coefficients related to Pascal's triangle?

**Example 1: Expand Binomials**

- Use Pascal's triangle to expand  $(a + b)^7$ .
- Identify patterns in the expansion of  $(a + b)^7$ .

**Solution:**

a.

b.

**THE BINOMIAL THEOREM**

If we wanted to expand a binomial expression with a large power, e.g.  $(x + y)^{32}$ , the use of Pascal's triangle would not be recommended because of the need to generate a large number of rows of the triangle. An alternative method is to use the **binomial theorem**.

**THE BINOMIAL THEOREM**

$$(x + y)^n = {}_n C_0 (x)^n (y)^0 + {}_n C_1 (x)^{n-1} (y)^1 + {}_n C_2 (x)^{n-2} (y)^2 + \dots + {}_n C_{n-1} (x)^1 (y)^{n-1} + {}_n C_n (x)^0 (y)^n$$

Each term has the form  ${}_n C_k (x)^{n-k} (y)^k$ , where  $k+1$  is the term number

**Explore the expansion presented below.**

$$(2a + 5)^4 =$$

$$\text{First term } {}_4 C_0 (2a)^4 (5)^0 = 1(16a^4) = 16a^4$$

$$\text{Second term } {}_4 C_1 (2a)^3 (5)^1 = 4(8a^3)(5) = 160a^3$$

$$\text{Third term } {}_4 C_2 (2a)^2 (5)^2 = 6(4a^2)(25) = 600a^2$$

$$\text{Fourth term } {}_4 C_3 (2a)^1 (5)^3 = 4(2a)(125) = 1000a$$

$$\text{Fifth term } {}_4 C_4 (2a)^0 (5)^4 = 1(0)(625) = 625$$

**Full expansion:** \_\_\_\_\_

**Determining a specific term in a binomial expansion.**

Find the eighth term in the expansion of  $(x + y)^{13}$ .

The first term of the expansion would be  ${}_{13} C_0 x^{13} y^0$ . The eighth term in the expansion would be  ${}_{13} C_7 x^6 y^7$

$${}_{13} C_7 = \frac{13!}{7!(13-7)!} = 1716$$

**Example 2: Use the Binomial Theorem**

- a. Use the binomial theorem to expand  $(3a - 4b)^5$ .
- b. What is the third term in the expansion of  $(5a^2 + 2)^6$ .
- c. In the expansion of  $\left(a^2 - \frac{2}{a}\right)^7$ , which term, in simplified form, contains  $a^5$ ?

**Solution:**

a.  $(3a - 4b)^5$

b. Third term in the expansion of  $(5a^2 + 2)^6$ .

c.  $\left(a^2 - \frac{2}{a}\right)^7$