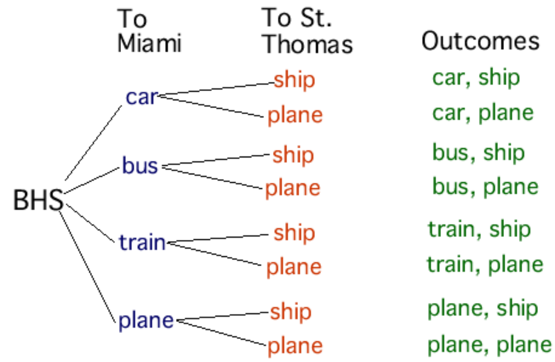
Permutations

Many questions in mathematics involve *counting*. For example, in how many ways can a committee of two men and three women be chosen from a group of 35 men and 40 women? How many different license plates can be made using three letters followed by three digits? How many different poker hands are possible?

Counting methods are used to determine the number of outcomes of an event. A tree diagram, for example, is one such way. A *tree diagram* is a graphic organizer used to list all possibilities of a sequence of events in a systematic way.



Example:

The senior class is planning a trip to the Virgin Islands. The class officers are exploring various travel options. From the high school, they will travel to Miami either by car, bus, train or plane. To travel to St. Thomas from Miami, they can take a plane or a ship. A tree diagram clearly illustrates each of the 8 possible ways the group can travel on their trip.

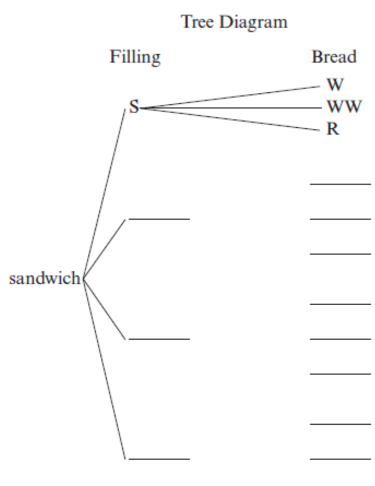
Example 1: Arrangements Without Restrictions

A school cafeteria offers sandwiches made with fillings of salami (S), ham (H), cheese (C), or egg (E), all of which are available on white (W), whole wheat (WW) or rye bread (R). How many different sandwiches can be made using only one filling?

Solution:

Method 1: List Outcomes and Count the Total

a. Use a tree diagram, or list all of the sandwich choices in a table, and count the outcomes.



List

Filling	Bread
S	W

______ possibilites

Method 2: Fundamental Counting Principle

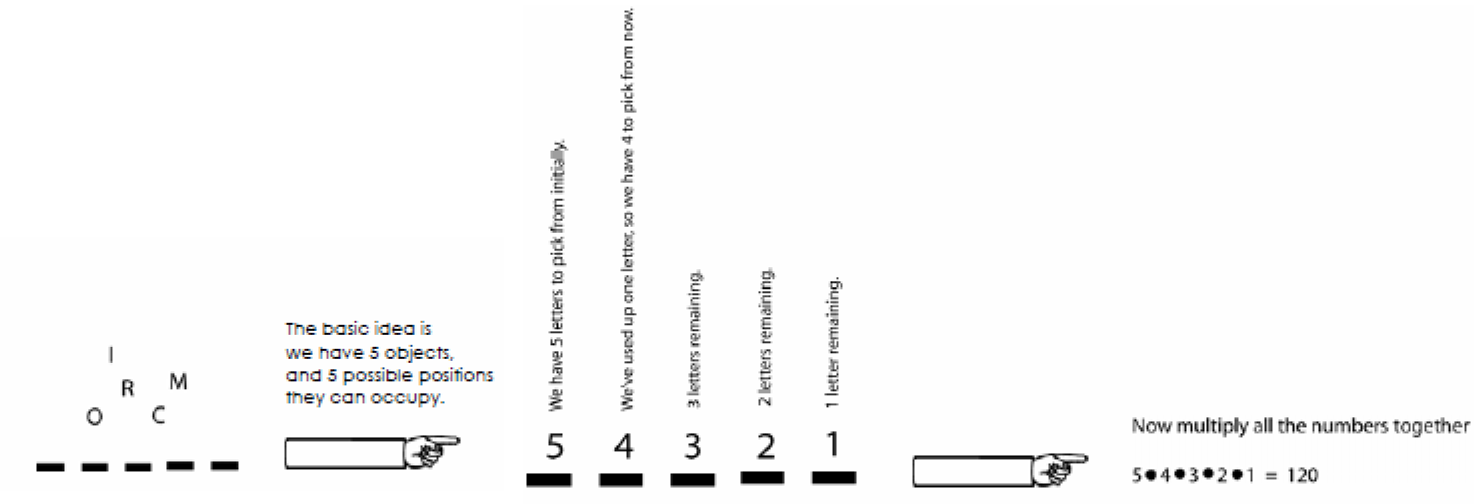
________________ x ________________ = ________________

(choices for sandwich filling) (number of choices for bread) (number of different sandwiches)

FUNDAMENTAL COUNTING PRINCIPLE

If one task can be performed in a ways and a second task can be performed in b ways, then the two tasks can be performed in a x b ways.

For example: How many ways can you arrange the letters in the word MICRO?



Example 2: Arrangements With Restrictions

In how many ways can five black cars and four red cars be parked next to each other in a parking garage if a black car has to be first and a red car has to be last?

Solution:

Whenever you encounter a situation with constraints or restrictions, always address the choices for the restricted positions first.

Use 9 blanks to represent the nine cars parked in a row. A black car must be in the first position and a red car must be in the last position. Fill these positions first.

_____ _____ _____ _____ _____ _____ _____ _____ _____

There are _______black cars for the first position. There are ________ red cars for the last position.
After filling the end positions, there are ______ positions to fill with ______________ cars remaining.
Use these numbers to fill in the blanks that represent the nine cars parked in a row.

By the fundamental counting principle, there are

(______)(______)(______)(______)(______)(______)(______)(______)(______)=______ ways to park the cars in a row.

FACTORIAL NOTATION:

Suppose you have 5 slips of paper, each containing a different test question. You are asked to make a test with 5 questions. In how many ways can you arrange the questions?

By the fundamental counting principle, there are $5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 120$ ways to arrange the questions.

The shorthand way to express this product is 5!

Factorial: for any positive integer, n, the product of all the positive integers up to and including n.

In general n!=(n)(n-1)(n-2)(n-3)(n-4)....(3)(2)(1)

For example, 8! = (8)(7)(6)(5)(4)(3)(2)(1)

By definition, 0!=1

Example 3: Using Factorial Notation

a. Simplify and evaluate: i.) $\frac{8!}{5!}$ ii.) $\frac{10!}{(10-4)!}$ iii.) $\frac{6!}{2!(6-2)!}$ iv.) $\frac{n!}{(n-2)!}$

b. Show that $20! - 19! + 18! = (362)(18!)$

Solution:

a. i. $\frac{8!}{5!} =$ ii. $\frac{10!}{(10-4)!} =$ iii. $\frac{6!}{2!(6-2)!}$ iv. $\frac{n!}{(n-2)!}$

b. Show that $20! - 19! + 18! = (362)(18!)$

PERMUTATIONS

Permutation is an arrangement in which a number of quantities are chosen with attention being paid to the order of choice. Examples: batting order for baseball team, license plate numbers, finish order of a race, PIN number for a bank account.

The combination to the safe was 472. We **do** care about the order of the digits. "724" would not work, nor would "247". It have to be exactly **4-7-2**.

Example 4: Formula for Permutations

Suppose there are eight runners in the final of a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties.

Solution:

Using the fundamental counting principle, there are (_______)(_______)(_______)= _______ possible ways to award the medals. This product can be written using factorials:

$$8\times7\times6 = \frac{8!}{5!} = \frac{8!}{(8-3)!} = {}_8P_3$$

The notation, ${}_nP_r$ is used to represent the number of permutations, or arrangement in a definite order, or r items taken from a set of n distinct items.

Formula for Permutations: ${}_nP_r = \frac{n!}{(n-r)!},\ n \in N$

Example 5: Working With Permutations 1

How many ways can you arrange all of the letters in the word MATH?

Solution:

Method 1: Fundamental Counting Principle

Method 2: Formula for Permutations

Example 6: Working With Permutations 2

a. How many 3-digit numbers can be created from the digits 5, 6, 3, 4, 7, 1, 2 without repeating any?
b. If there are 40 clarinet players competing for places in the New Brunswick Youth Orchestra, how many ways can the 1st and 2nd chairs be filled?

Solution:

a.

b.

Example 7: Permutations with Repeating Objects – Repetitions Not Allowed

How many distinct ways can you arrange all of the letters in the "word" SHHH?

If we proceed as we did in the MATH problem (Example 5) above, we would get a total of 4! possibilities. Unfortunately, this method over-counts since three of the letters are the same, so we need to correct for this. But how?

Let's pretend that the three H's are different, and call them H_1, H_2, and H_3. The 4! = 24 arrangements of the letters in SHHH would look like this:

$$SH_1H_2H_3,\ SH_1H_3H_2,\ SH_2H_1H_3,\ SH_2H_3H_1,\ SH_3H_1H_2,\ SH_3H_2H_1\ \Rightarrow\ SHHH$$
$$H_1SH_2H_3,\ H_1SH_3H_2,\ H_2SH_1H_3,\ H_2SH_3H_1,\ H_3SH_1H_2,\ H_3SH_2H_1\ \Rightarrow\ HSHH$$
$$H_1H_2SH_3,\ H_1H_3SH_2,\ H_2H_1SH_3,\ H_2H_3SH_1,\ H_3H_1SH_2,\ H_3H_2SH_1\ \Rightarrow\ HHSH$$
$$H_1H_2H_3S,\ H_1H_3H_2S,\ H_2H_1H_3S,\ H_2H_3H_1S,\ H_3H_1H_2S,\ H_3H_2H_1S\ \Rightarrow\ HHHS$$

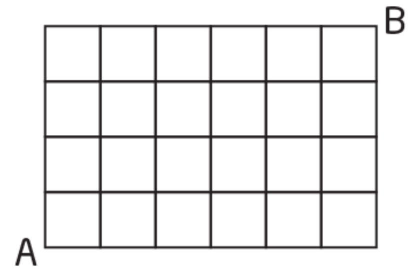
The 24 arrangements are really just 4 *distinct* arrangements.

So the number of distinct arrangements of the letters in the word SHHH is $\frac{4!}{3!}=\frac{24}{6}=4$

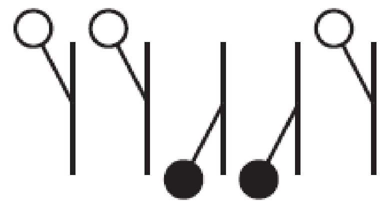
Example 8: Permutations With Repeating Objects- Repetitions Not Allowed

a. How many different twelve-letter arrangements can you make using the letters of NEWFOUNDLAND?

b. How many paths can you follow from A to B in a four by six rectangular grid if you move only up or to the right?



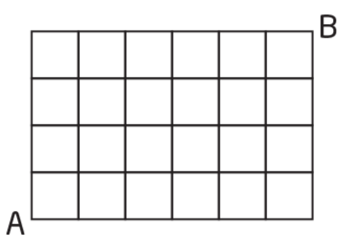
c. An electrical panel has five switches. How many ways can the switches be positioned up or down if three switches must be up and two must be down.



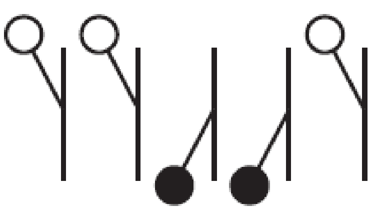
Solution:

a. NEWFOUNDLAND?

b.



c.



Example 9: Permutations Where Repetitions Are Allowed

A phone number in British Columbia consists of one of four area codes (236, 250, 604, and 778), followed by a 7-digit number that cannot begin with a 0 or 1. How many unique numbers are there?

Solution:

Example 10: Permutations With Constraints

Nick, Hye Won, Melissa, Marina, Miranda and Jacob are seated in the front row in Mrs. Power's Pre-Calculus 12 classroom. In how many ways can they be arranged if

a. Melissa must be seated at the third desk?
b. Jacob can't be at either end of the line?
c. the row starts with exactly two females?
d. Nick, Hye Won and Miranda must be seated together?
e. Nick and Marina cannot be seated together?

Solution:

a. Melissa must be seated at the third desk?

b. Jacob can't be at either end of the line?

c. The row starts with exactly two females?

d. Nick, Hye Won and Miranda must be seated together?

e. Nick and Marina cannot be seated together?

ARRANGEMENTS REQUIRING CASES

To solve some problems, you must count the different arrangements in all the cases that together cover all the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.

Example 11: Using Cases to Determine Permutations 1

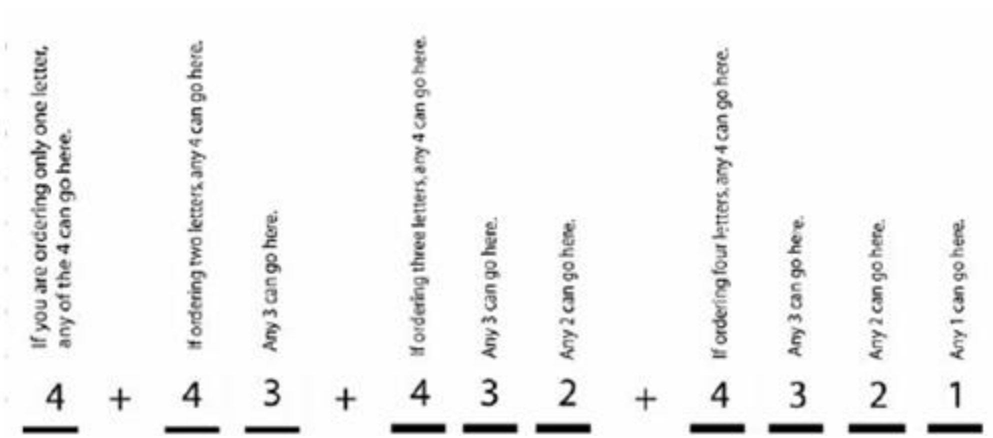
a. How many "words" (of any number of letters) can be formed from the letters C A N S?

b. How many "words" (with at least five letters) can be formed from the letters tiles S U N D A Y?

Solution:

There are four cases to be considered: 1 letter words, 2 letter words, 3 letter words, and 4 letter words.

a. Case 1 Case 2 Case 3 Case 4



We could also write this using permutations: ________ + ________ + ________ + ________ = ________

b. S U N D A Y

Example 12: Using Cases to Determine Permutations 2

How many four digit positive numbers less than 4670 can be formed using only the digits 1, 3, 4, 5, 8, 9 if repetitions are not allowed.

Solution:

Case 1: Numbers in the 4000's

There is only one possibility for the first digit (4). The second digit has three possibilities (1, 3, 5). Now there are four possibilities for the next digit since any remaining number can be used, and three possibilities for the last digit.

Case 2: Numbers in the 1000's and 3000's

There are two possibilities for the first digit (1,3). Then the remaining digits would have five possibilities, then four and finally three.

Total the result:

Example 13: Arrangements Requiring Cases

In how many ways can eight basketball players sit on a bench if either the one centre or both of the two forwards must sit at the end where the coach always sits?

Solution:

Case 1: The centre sits beside the coach

Case 2: The two forwards sit beside the coach

Total number of seating arrangements=

Example 14: Solving Equations with Factorials & Permutations

Solve

a. $\frac{n!}{10} = {}_{n-1}P_{n-3}$

b. ${}_{n+3}P_2 = 20$

Solution:

a. $\frac{n!}{10} = {}_{n-1}P_{n-3}$

b. ${}_{n+3}P_2 = 20$