Limits (Review)

- definition of limit
- properties of limits
- one-sided and two-sided limits
- Sandwich Theorem

Definition of a Limit

- most limits can be viewed as numerical limits of values of functions
- calculators can suggest the limits
- calculus allows us to confirm the limits analytically

DEFINITION Limit

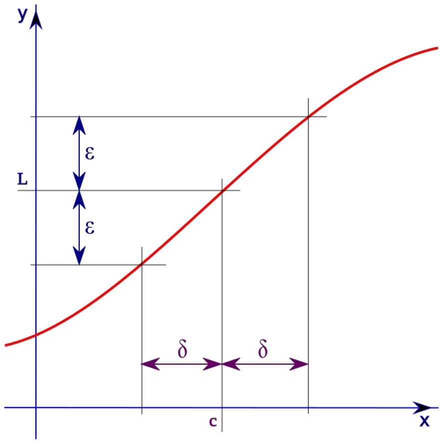
Assume f is defined in a neighborhood of c and let c and L be real numbers. The function ***f*** **has a limit** ***L*** **as** ***x*** **approaches** ***c*** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

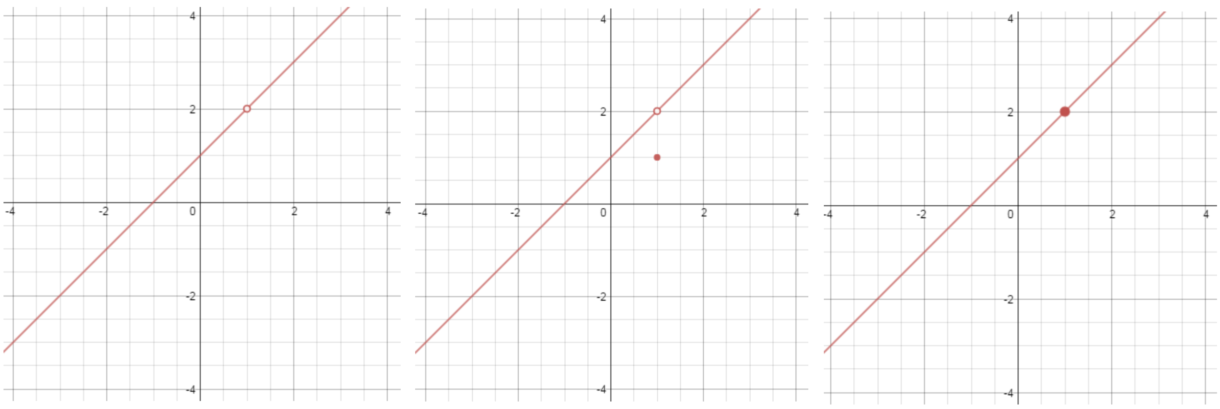
if the above is true, we write: $\lim_{x \to c} f(x) = L$

What does that even mean?!

$$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$$



Definition of a Limit



$$f(x)=\frac{x^2-1}{x-1}$$

$$g(x)\begin{cases}\dfrac{x^2-1}{x-1}, x\neq 1\\ 1, x=1\end{cases}$$

$$h(x)=x+1$$

Properties of Limits:

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know.

Basic Limits:

$$\lim_{x \to c}(k) = k \qquad \lim_{x \to c}(x) = c$$

The Properties of Limits Theorem shows us all the facts we can use when solving limits.

THEOREM Properties of Limits

If *L, M, a* and *c* are real numbers and $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = M$

1. *Sum Rule:* $\lim_{x \to a}\left[f(x) + g(x)\right] = L + M$
2. *Difference Rule:* $\lim_{x \to a}\left[f(x) - g(x)\right] = L - M$
3. *Product Rule:* $\lim_{x \to a}\left[f(x) \times g(x)\right] = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to a}\left[c \cdot f(x)\right] = c \cdot L$
5. *Quotient Rule:* $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}; \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to a}\left[f(x)\right]^{r/s} = L^{r/s}$

THEOREM Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}; \quad g(c) \neq 0$$

EXAMPLE 1:

Evaluate the following limits

a) $\lim_{x \to 3} \frac{\left(\frac{1}{x-2}\right) - 1}{x - 3}$

b) $\lim_{x \to 5} \frac{\sqrt{9 - x} - 2}{x - 5}$

c) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

One-Sided and Two-Sided Limits

- sometimes the values of a function tend to different limits as *x* approaches *c* from opposite sides
- **right-hand limit:** $\lim_{x \to c^+} f(x)$
- **left-hand limit:** $\lim_{x \to c^-} f(x)$

THEOREM One-Sided and Two-Sided Limits

A function *f*(*x*) has a limit as *x* approaches *c* **if and only if** the right-hand and left-hand limits at *c* exist and are equal.

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L$$

EXAMPLE 2:

Evaluate $\lim_{x \to 6} \frac{|x-6|}{x-6}$

Practice Problems:
p. 66 # 1-4, 18, 19, 43, 46, 48, 55, 58, 63, 69, 70

Sandwich Theorem

- we can use the Sandwich Theorem to calculate limits
- we can use it when a function's values are *sandwiched* between the values of two other functions

THEOREM The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \neq c$ in some interval near c, and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} f(x) = L$

EXAMPLE 3: Using the Sandwich Theorem
Determine $\lim_{x \to 1} f(x)$ if $-4x \le f(x) \le x^2 - 6x + 1$

EXAMPLE 4: Using the Sandwich Theorem
Use the graph to solve $\lim_{x \to 3} f(x)$