Derivative of a Composite Function:
How would we find the derivative of
- $y = (x + 1)^{10}$
- $y = \sin(x + 2)$

Consider the function $y = \sin(x^2 - 4)$.

So far, we have rules for all types of combinations of functions (sum, difference, product, quotient) but not compositions. For composite functions we use the ***Chain Rule***.

EXAMPLE 1: Relating Derivatives
The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three related?

EXAMPLE 2: Relating Derivatives
The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. How are the derivatives of these three functions related?

The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

where dy/du is evaluated at $u = g(x)$.

<u>EXAMPLE 3:</u> Applying the Chain Rule

An object moves along the s-axis so that its position at any time $t \geq 0$ is given by $s(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

"Outside-Inside" Rule
It sometimes helps to think about about the chain rule this way:

> differentiate the **outside** function f and evaluate it
> at the **inside** function $g(x)$ left alone; then multiply
> by the derivative of the **inside function**

<u>EXAMPLE 4:</u> Differentiating from the Outside In
Differentiate $\sin(x^2 + x)$ with respect to x.

Repeated Use of the Chain Rule
Sometimes we need to use the Chain Rule two or more times to find a derivative.

<u>EXAMPLE 5:</u> A Three-Link "Chain"
Find the derivative of $g(t) = \tan[5 - \sin(2t)]$

Power Chain Rule

If n is an integer and $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}u^n = nu^{n-1}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1}\cdot f'(x)$$

<u>**EXAMPLE 6:**</u> **Finding Slope**

a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \frac{\pi}{3}$

b) Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.